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Perturbation of an eigenvalue from a dense point spectrum: an example

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Abstract. We study an example of a perturbed Floquet Hamiltonian $K + \beta V$ depending on a coupling constant β. The spectrum $\sigma(K)$ is pure point and dense. We pick up an eigenvalue, namely $0 \in \sigma(K)$, and show the existence of a function $\lambda(\beta)$ defined on $I \subset \mathbb{R}$ such that $\lambda(\beta) \in \sigma(K + \beta V)$ for all $\beta \in I$, 0 is a point of density for the set I, and the Rayleigh–Schrödinger perturbation series represents an asymptotic series for the function $\lambda(\beta)$. All ideas are developed and demonstrated when treating the explicit example, but some of them are expected to have an essentially wider range of application.

1. Introduction

A common problem occurring frequently in theoretical physics is the eigenvalue problem for a perturbed operator $K + \beta V$, where β is a coupling constant, under the assumption that F_0 is a known eigenvalue of the unperturbed operator K. The Rayleigh–Schrödinger (RS) series gives a formal solution $F(\beta)$, with $F(0) = F_0$, as an unambiguously determined formal power series. The regular perturbation theory due to Rellich (1937) and Kato (1966) justifies this formal series as an analytic function well defined on a neighbourhood of $\beta = 0$ provided one essential condition is fulfilled, namely the eigenvalue $F_0 \in \sigma(K)$ must be *isolated*. On the other hand, the situation when an eigenvalue of K is not isolated is far from being exceptional and recently attracted considerable attention (see Simon 1993 and references therein).

The so-called Floquet Hamiltonians represent a class of operators having even a dense pure point spectrum in many interesting examples. They were introduced as an important tool for the study of time-dependent systems (see, e.g., Sambe 1973, Howland 1979, Yajima 1977). A distinguished subclass is formed by the systems where the potential $V(t)$ is T-periodic and bounded. The period is usually considered as a parameter. After rescaling the time, the potential $V(t)$ becomes 2π-periodic and the frequency $\omega = 2\pi/T$ appears in

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front of the time derivative. Thus one is lead to study the operator $K + \beta V(t)$ acting in $\mathcal{K} := L^2(\mathbb{T}, dt) \otimes \mathcal{H}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and

$$K := -\mathrm{i}\omega\partial_t + H \qquad \omega > 0$$

where H is the 'true' Hamiltonian acting as a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. We use the loose notation identifying ∂_t with $\partial_t \otimes 1$, H with $1 \otimes H$, etc. Provided the spectrum $\sigma(H)$ is pure point the same is true for $\sigma(K) = \omega\mathbb{Z} + \sigma(H)$. It is known that $\sigma(K)$ is dense in $\mathbb{R}$ for almost all $\omega > 0$ as soon as $\sup\sigma(H) = +\infty$. Recently the spectrum of $K + \beta V(t)$ has been studied by the aid of a quantum version of the KAM method due to Bellissard (1985) (see also Combescure 1987, Bellissard and Vittot 1990, Bleher, Jauslin and Lebowitz 1992, Duclos and Šťovíček 1996a) as well as by adiabatic tools (Howland 1989, 1992, Nenciu 1993, Joye 1994).

In the present paper we focus on a particular example with $\mathcal{H} = L^2(\mathbb{T}, dx)$, namely

$$H = -\partial_x^2 \text{ (+ periodic boundary conditions)} \qquad V(t) = 4\cos t \cos x. \tag{1}$$

Clearly, $\sigma(H) = \{E(k) = k^2;\ k \in \mathbb{Z}\}$ and so $\sigma(K) = \{F(n) = \omega n_1 + E(n_2);\ n \in \mathbb{Z} \times \mathbb{Z}\}$. The spectrum of H is degenerate and that makes the problem more complicated; the only non-degenerate eigenvalue is $E(0) = 0$. This is why we restrict ourselves to eigenvalues $F(n)$ of K with $n_2 = 0$. In order to be specific, we shall even consider the only eigenvalue $F(0) = 0$. We are going to address the question whether there exists an eigenvalue $\lambda(\beta)$ of the operator $K + \beta V(t)$ which could be considered as a perturbation of $F(0) = 0$ depending on the parameter β. A possible answer is given in ($|X|$ stands for the Lebesgue measure of a measurable set X)

Proposition 1. For almost all $\omega > 0$, there exists a real-valued function $\lambda(\beta)$ defined on a set $I \subset \mathbb{R}$ with the properties:

(i) for all $\beta \in I$, $\lambda(\beta)$ is an eigenvalue of $K + \beta V(t)$,
(ii) $\lim_{\delta\downarrow 0} |I \cap [-\delta, \delta]|/2\delta = 1$,
(iii) the function $\lambda(\beta)$ has an asymptotic expansion at $\beta = 0$ coinciding with the formal Rayleigh–Schrödinger perturbation series for the eigenvalue $F(0) = 0$ of K.

In fact, our final goal (not achieved in this paper) is to prove a similar proposition for a much wider class of Floquet Hamiltonians. More precisely we expect that statements (i) and (ii) of proposition 1 remain valid in the following more general situation: (a) the Hilbert space $\mathcal{H}$ is separable but otherwise unspecified, (b) the spectrum of H consists only of eigenvalues $\sigma(H) = \{E(k);\ k \in \mathbb{Z}_+\}$ with uniformly finite multiplicity and fulfill the gap condition:

$$\exists\alpha > 0 \qquad \inf_{k\in\mathbb{Z}_+} \frac{E(k+1) - E(k)}{(k+1)^\alpha} =: C_E > 0 \tag{2}$$

(note that $\alpha = 1$ in our example), (c) V is $[\frac{2}{\alpha}] + 1$ strongly time differentiable as a map from $\mathbb{T}$ into the bounded operators in $\mathcal{H}$ (note that here as well as in what follows $[x]$ denotes the integer part of x), and (d) the unperturbed eigenvalue is any of the eigenvalues of $K = -\mathrm{i}\omega\partial_t + H$. In addition we think that the Rayleigh–Schrödinger perturbation series still exist up to an order depending on the regularity of V.

These more general conditions cover a large class of quantum driven systems as, e.g., the pulsed rotor (our example), Bellissard 1985, the quantum Fermi accelerator (Fermi 1949, Ulam 1961, Duclos and Šťovíček 1996b) as well as other quantum systems like a Bloch electron driven by a constant electric force in singular periodic potentials (Ao 1989, Avron *et al* 1994). The relevance of such models to study the mechanism behind the phenomena of dynamical stability and localisation is well known.

Condition (2) on the growth of the gaps in the spectrum of H was put forward in Howland 1989 and is used explicitely or implicitely in most of the known results quoted above. However, Combescure (1987) considered the case $\alpha = 0$ with an extra assumption, leading to non-realistic potential. One would even like to treat negative α, as in the famous 'quantum ball' model (see Benvenuto *et al* 1991). We think that the study of such more difficult cases will require 'compensation techniques' of the type which is used in lemma 10 below. This is also one of the purposes of this paper, namely to get some training in using such a new tool. Note that we learned such techniques in Eliasson 1988.

As this programme seems to be extremely complex, we prefer to develop and demonstrate the main ideas when treating the explicit example of the pulsed rotor. Even in this example the proof is far from obvious and straightforward. Apparently, our model captures most of the basic features but, on the other hand, it makes some simplifications possible and can be treated at a relatively elementary level. In particular, our model is very regular in the following sense: writing $\{V(n;m)\}$ for the matrix elements of the potential V in the eigenbasis of K we have that: $\exists C > 0$, $\forall r \geqslant 0$, $|V(n,m)| \leqslant C|n-m|^{-r}$ (one has even exponential bounds), a property which is not true, e.g., for the quantum Fermi accelerators or the Bloch electron quoted above. On the other hand, the pulsed rotor is not the simplest choice of example since the spectrum of H is two-fold degenerate as already mentioned above. For example, we could have taken $H = -\Delta$ on a compact interval with Dirichlet boundary conditions or $H = -\Delta$ on $\mathbb{T}$ with twisted boundary conditions. We prefer to stick to this example, since we believe that it is more attractive due to its historical value (see Bellissard 1985).

The rest of the paper is devoted to the proof of proposition 1. However, we shall try, whenever possible, to consider a more general situation and to propose some ideas that are also applicable to other models. It might have been possible to follow all the constants in the proof of proposition 1 and give an explicit bound on the size of I, as well as an explicit lower bound on the measure of $|I \cap [-\delta, \delta]|$ with the risk of making the derivation less transparent. We hope to come back to such a challenge when treating the general situation.

2. Basic equation

The starting point is the eigenvalue equation for $K+\beta V$. Assume that 0 is a non-degenerate eigenvalue of K and f is the normalized eigenvector. Let P be the orthogonal projector onto the eigenspace $\mathbb{C}f$ and $Q := 1 - P$. We are seeking $\lambda = \lambda(\beta) \in \mathbb{R}$ and $g \in \mathcal{K}$ such that $Pg = 0$ and

$$(K + \beta V)(f + g) = \lambda(f + g). \tag{3}$$

Without loss of generality we can assume that

$$PVP = 0. \tag{4}$$

Apply the projectors P and Q successively to equation (3). The result is

$$\lambda = \beta \langle Vf,\, g\rangle \tag{5}$$

$$(\hat{K} + \beta\hat{V} - \lambda)g = -\beta QVf. \tag{6}$$

Here and everywhere in what follows the hat indicates the restriction to Ran Q in the sense: $\hat{X} = QXQ|$ Ran Q.

According to our assumptions, $\hat{K}$ is invertible and we set $\Gamma_0 := \hat{K}^{-1}$ (defined on Ran Q). For $\lambda \notin \sigma(\hat{K})$ we also define

$$\Gamma_\lambda := (\hat{K} - \lambda)^{-1} = (1 - \lambda\Gamma_0)^{-1}\Gamma_0.$$

Keeping λ as an auxiliary parameter one can solve (6) formally as

$$g = g(\beta, \lambda) := -\beta(1 + \beta\Gamma_\lambda\hat{V})^{-1}\Gamma_\lambda QVf. \tag{7}$$

Inserting (7) in (5) we obtain a fixed-point equation for the eigenvalue $\lambda = \lambda(\beta)$:

$$\lambda = G(\beta, \lambda) \qquad \text{where} \quad G(\beta, \lambda) := -\beta^2\langle QVf,\ (1 + \beta\Gamma_\lambda\hat{V})^{-1}\Gamma_\lambda QVf\rangle. \tag{8}$$

The trick with the projectors and keeping λ as an auxiliary parameter is well known and is related to various names. Note that in the regular case, when $d := \mathrm{dist}(0, \sigma(\hat{K})) > 0$, one can re-derive the Rellich–Kato theorem in this way. Indeed, we have $\|\Gamma_0\| = d^{-1}$ and $(1 + \beta\Gamma_\lambda\hat{V})$ is invertible (on Ran Q) provided $|\beta|$ and $|\lambda|$ are sufficiently small. The implicit function theorem applied to (8) then gives the result.

To solve (8) formally one can use Bürmann–Lagrange formula which can be proved with some combinatorics, and not necessarily with the Cauchy residuum theorem. Write

$$G(\beta, \lambda) = \sum_{M=0}^{\infty} \Phi_M(\beta)\,\lambda^M$$

where

$$\Phi_M(\beta) = -\sum_{k=1}^{\infty}\ \sum_{\mu\in\mathbb{N}^k,\ |\mu|=k+M} (-\beta)^{k+1}\,\langle QVf,\ \hat{K}^{-\mu_1}\hat{V}\hat{K}^{-\mu_2}\cdots\hat{V}\hat{K}^{-\mu_k}QVf\rangle.$$

The formal solution $\lambda(\beta)$ reads

$$\lambda(\beta) = \sum_{N=1}^{\infty}\sum_{\nu\in\mathcal{T}(N)} \Phi_{\nu_1}(\beta)\cdots\Phi_{\nu_N}(\beta) = \sum_{M=2}^{\infty}\xi_M\,\beta^M \tag{9}$$

where $\mathcal{T}(N) \subset \mathbb{Z}_+^N$ is the set of rooted N-trees: $\nu = (\nu_1, \ldots, \nu_N) \in \mathcal{T}(N)$ iff $\nu_k + \cdots + \nu_N \leqslant N - k$, $2 \leqslant k \leqslant N$, and $|\nu| = N - 1$. Consequently, one obtains an expression for the coefficients ξ_M:

$$\xi_M = \sum_{N=1}^{[M/2]}\sum_{\nu\in\mathcal{T}(N)}\ \sum_{k(1),\ldots,k(N)\in\mathbb{N}}\ \sum_{\mu(1)\in\mathbb{N}^{k(1)},\ldots,\mu(N)\in\mathbb{N}^{k(N)}}$$

$$\times\,(-1)^{M+N}\prod_{j=1}^{N}\langle QVf,\ \hat{K}^{-\mu(j)_1}\hat{V}\hat{K}^{-\mu(j)_2}\cdots\hat{V}\hat{K}^{-\mu(j)_{k(j)}}QVf\rangle \tag{10}$$

with the summation range being restricted by the conditions

$$k(1) + \cdots + k(N) + N = M \qquad |\mu(j)| = k(j) + \nu_j \quad 1 \leqslant j \leqslant N.$$

Of course, this result must coincide with the standard RS perturbation series written in the form (see Kato 1966):

$$\xi_M = \frac{(-1)^M}{M}\sum_{k_1+\cdots+k_M=M-1,\ k_i\geqslant 0}\mathrm{tr}\left(V\hat{R}^{k_1}\cdots V\hat{R}^{k_M}\right) \tag{11}$$

where the symbol $\hat{R}^k$ is defined by: $\hat{R}^0 = -P$, and for $k \geqslant 1$, $\hat{R}^k|\,\mathrm{Ran}\,P = 0$, $\hat{R}^k|\,\mathrm{Ran}\,Q = \hat{K}^{-k}$. The equality between (10) and (11) can be verified quite straightforwardly using (4) and the following fact.

Lemma 2. For a given $N \in \mathbb{N}$ and each $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{Z}_+^N$ obeying $|\sigma| = N - 1$ there exists exactly one cyclic permutation of σ, $\sigma' = (\sigma_{N-m+1}, \ldots, \sigma_N, \sigma_1, \ldots, \sigma_{N-m})$ (determined by $m \in \{0, 1, \ldots, N-1\}$), such that $\sigma' \in \mathcal{T}(N)$.

Hence each term of (10) is a grouping of many terms of (11) where we take into account the cyclic property of the trace.

However, in the case when $\sigma(K)$ is dense in $\mathbb{R}$ and so $\operatorname{dist}(0, \sigma(\hat{K})) = 0$ it seems to be hopeless to consider the RS series as a convergent series. The complication comes from arbitrarily large powers of $\hat{K}^{-1}$ in (10) (or (11)) since among eigenvalues of $\hat{K}$ there are arbitrarily small numbers, the so-called small denominators. Probably the maximum one can attempt in this situation is to verify the finiteness of the coefficients ξ_M (generally up to some order depending on the smoothness of $V(t)$) and to show that the RS series is asymptotic for the function $\lambda(\beta)$.

Let us specify the formula (10) in our example (1). Consider $V(t)$ as an operator in $\mathcal{K}$ and denote by $V(m,n)$, $m, n \in \mathbb{Z}^2$, its matrix elements in the eigenbasis of K. We have

$$V(m,n) = \begin{cases} 1 & \text{if } m - n \in \{\pm(1,1),\ \pm(1,-1)\} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Concerning the eigenvalues of K, there is a degeneracy

$$F(n_1, n_2) = F(n_1, -n_2) = \omega n_1 + n_2^2.$$

Let $\mathbb{L} = \mathbb{Z}(1,1) + \mathbb{Z}(1,-1)$ be a sublattice in $\mathbb{Z}^2$ and denote by $\mathcal{P}_0(N) \subset (\mathbb{Z}^2)^{N+1}$ the set of closed paths in $\mathbb{L}$ of length N with the base point $\bar{0}$: $(\bar{\iota}(0), \bar{\iota}(1), \ldots, \bar{\iota}(N)) \in \mathcal{P}_0(N)$ iff $\bar{\iota}(0) = \bar{\iota}(N) = \bar{0}$, $\bar{\iota}(j) \neq \bar{0}$ for $1 \leqslant j \leqslant N-1$, $\bar{\iota}(j) - \bar{\iota}(j-1) \in \{\pm(1,1),\ \pm(1,-1)\}$ for $1 \leqslant j \leqslant N$. Note that $\mathcal{P}_0(N) = \emptyset$ for N odd. Clearly

$$\langle QVf,\ \hat{K}^{-\mu_1} \hat{V} \hat{K}^{-\mu_2} \cdots \hat{V} \hat{K}^{-\mu_k} QVf \rangle = \sum_{\bar{\iota} \in \mathcal{P}_0(k+1)} \prod_{j=1}^{k} F(\bar{\iota}(j))^{-\mu_j}. \tag{13}$$

The only thing we can claim at this moment is that all ξ_M, $2 \leqslant M$, are finite for the sum on the right-hand side of (10) is finite.

3. Diophantine estimates

In order to cope with small denominators we need diophantine estimates. Suppose that we are given two sequences ψ and E such that

$$\psi : \mathbb{N} \to \left]0, \tfrac{1}{2}\right] \qquad \sum_{k \in \mathbb{N}} \psi(k) < \infty$$

and

$$E : \mathbb{N} \to]0, +\infty[\qquad \inf_{k \in \mathbb{N}} E(k) =: d_E > 0.$$

Set $F(n) := \omega n_1 + E(n_2)$, $n \in \mathbb{Z} \times \mathbb{N}$, and relate the set

$$\Omega(\gamma) := \{\omega > 0;\ \forall n \in \mathbb{Z} \times \mathbb{N},\ |F(n)| \geqslant \omega \gamma \psi(n_2)\}$$

to a constant $\gamma > 0$. It is a rather standard procedure to show

Lemma 3. If $\gamma \leqslant d_E / a \leqslant 1$ then

$$|\,]0, a] \setminus \Omega(\gamma)| \leqslant \left(12a \sum_{k \in \mathbb{N}} \psi(k) \right) \gamma.$$

Proof. Write

$$]0,a\,]\setminus\Omega(\gamma)=\bigcup_{n\in\mathbb{Z}\times\mathbb{N}}\Omega_{\mathrm{bad}}(n)$$

where

$$\Omega_{\mathrm{bad}}(n):=\{\omega\in\,]0,a];\;\;|F(n)|<\omega\gamma\;\psi(n_2)\}.$$

Consider $n\in\mathbb{Z}\times\mathbb{N}$ such that $\Omega_{\mathrm{bad}}(n)\neq\emptyset$. Clearly

$$\omega\in\Omega_{\mathrm{bad}}(n)\Rightarrow E(n_2)-\omega\gamma\psi(n_2)<|\omega n_1|<E(n_2)+\omega\gamma\psi(n_2).$$

We shall need two consequences of these inequalities. First, since

$$\omega\gamma\psi(n_2)\leqslant d_E/2\leqslant E(n_2)/2$$

we obtain

$$0<E(n_2)/2a<|n_1|.$$

Second, as we now know that $|n_1|\geqslant 1>\gamma\psi(n_2)$, we have

$$\frac{E(n_2)}{|n_1|+\gamma\psi(n_2)}<\omega<\frac{E(n_2)}{|n_1|-\gamma\psi(n_2)}.$$

Consequently

$$|\Omega_{\mathrm{bad}}(n)|\leqslant\frac{2\gamma\psi(n_2)E(n_2)}{n_1^2-\gamma^2\psi(n_2)^2}<\frac{3\gamma\psi(n_2)E(n_2)}{n_1^2}.$$

Furthermore, $n_1<0$ since otherwise $n_1\geqslant 1$ and

$$F(n)=\omega n_1+E(n_2)>\omega>\omega\gamma\psi(n_2)$$

for all $\omega\in\Omega_{\mathrm{bad}}(n)\neq\emptyset$, a contradiction. We conclude that

$$|\;]0,a\,]\setminus\Omega(\gamma)|\leqslant\sum_{n_2\in\mathbb{N}}3\gamma\;\psi(n_2)\;E(n_2)\quad\sum_{\substack{n_1\in\mathbb{N}\\ n_1>E(n_2)/2a}}\frac{1}{n_1^2}.$$

To complete the proof observe that, for $x>0$,

$$\sum_{k\in\mathbb{N},\;k>x}\frac{1}{k^2}\leqslant\frac{2}{x}.$$

□

We can now introduce the set Ω (depending on ψ) of 'non-resonant' frequencies:

$$\Omega:=\{\omega>0;\;\inf_{n\in\mathbb{Z}\times\mathbb{N}}|F(n)|/\psi(n_2)>0\}=\bigcup_{\gamma>0}\Omega(\gamma).$$

As an immediate consequence of lemma 3 we have

Lemma 4. The complement $]0,+\infty[\,\setminus\Omega$ is of zero measure in the Lebesgue sense.

In the case of our model, $E(k)=k^2$. Extend the definition of ψ by $\psi(0)=1$ and we define also $F((k,0)):=\omega k$. We *fix* $\omega\in\Omega$ once for all (and we do not emphasize this fact in the rest of the paper). Then there exists $\gamma,\;0<\gamma\leqslant 1$, such that

$$|F(n)|\geqslant\omega\gamma\;\psi(n_2)\qquad\text{for all }\;n\in\mathbb{Z}\times\mathbb{Z}_+\quad n\neq\bar{0}.$$

Rather than treating the formal RS series (9), we wish to tackle the fixed-point equation (8). This means coping with expressions involving the operator Γ_λ and hence the numbers $(F(n)-\lambda)^{-1}$, i.e. the eigenvalues of Γ_λ. The estimate on $F(n)-\lambda$ will be governed by a constant ρ and a sequence $\tilde{\psi}$ of positive reals and we require

$$\rho \in [0,1] \qquad \text{and} \qquad \tilde{\psi}(k) \leqslant \psi(k)/2 \quad \forall\, k \in \mathbb{Z}_+.$$

For a given sequence E as above we define a set Λ of 'good' parameters λ:

$$\Lambda := \{\lambda \in \mathbb{R};\ \forall\, n \in \mathbb{Z}\times\mathbb{N},\ |F(n)-\lambda| \geqslant \omega\gamma\,(2|\lambda|/\omega)^\rho\,\tilde{\psi}(n_2)\}. \tag{14}$$

Note that $|F(n)-\lambda| \geqslant \omega/2$ if $n_1 \neq 0,\ n_2 = 0$ and $|\lambda| \leqslant \omega/2$. It is important for us to control the measure of Λ.

Lemma 5. If $0 < \delta \leqslant \frac{1}{4}$ then

$$|\,[-\delta\omega, \delta\omega\,] \setminus \Lambda| < 2\omega\gamma\,(2\delta)^\rho \sum_{k\in\mathbb{N},\ \psi(k)<2\delta/\gamma} \tilde{\psi}(k).$$

Proof. Fix $\delta,\ 0 < \delta \leqslant \frac{1}{4}$, and write

$$[-\delta\omega, \delta\omega] \setminus \Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_{\mathrm{bad}}(k)$$

where

$$\Lambda_{\mathrm{bad}}(k) := \left\{\lambda \in [-\delta\omega, \delta\omega];\ \min_{j\in\mathbb{Z}} |\omega j + E(k) - \lambda| < \omega\gamma\,(2|\lambda|/\omega)^\rho\,\tilde{\psi}(k)\right\}.$$

Observe that for a given $k \in \mathbb{N}$ there is at most one $j \in \mathbb{Z}$ such that there exists $\lambda \in [-\delta\omega, \delta\omega]$ for which $|\omega j + E(k) - \lambda| < \omega\gamma(2|\lambda|/\omega)^\rho\tilde{\psi}(k)$. Indeed, if another couple j', λ' were to exist then

$$\omega|j-j'| < 2\omega\gamma\,(2|\lambda|/\omega)^\rho\tilde{\psi}(k) + |\lambda| + |\lambda'| \leqslant \frac{\omega}{2} + \frac{\omega}{4} + \frac{\omega}{4} = \omega$$

and hence $j = j'$. It follows that

$$|\Lambda_{\mathrm{bad}}(k)| \leqslant 2\omega\gamma(2\delta)^\rho\tilde{\psi}(k).$$

Furthermore, if $\lambda \in \Lambda_{\mathrm{bad}}(k) \neq \emptyset$ then, since $\omega \in \Omega$,

$$\begin{aligned}\omega\gamma\psi(k) - |\lambda| &\leqslant |\omega j + E(k) - \lambda| < \tfrac{1}{2}\,\omega\gamma\,\psi(k)\\ &\Longrightarrow \tfrac{1}{2}\,\omega\gamma\,\psi(k) < |\lambda| \leqslant \delta\omega\\ &\Longrightarrow \psi(k) < 2\delta/\gamma.\end{aligned}$$

The assertion is then a direct consequence. □

The standard choice for ψ and $\tilde{\psi}$ is

$$\psi(k) = k^{-\sigma}/2 \qquad \tilde{\psi}(k) = k^{-\tau}/4 \qquad \text{with}\ \ 1 < \sigma \leqslant \tau. \tag{15}$$

In this case we obtain another intermediate result as a direct consequence of lemma 5.

Lemma 6. If $\tau > 1 + \sigma(1-\rho)$ then 0 is a point of density for the set Λ, i.e.

$$\lim_{\delta\downarrow 0} \frac{1}{2\delta\omega}\ |[-\delta\omega, \delta\omega\,] \cap \Lambda| = 1.$$

Proof. It is sufficient to estimate

$$\sum_{k\in\mathbb{N},\ k>(\gamma/4\delta)^{1/\sigma}} k^{-\tau} \leqslant \left(\frac{4\delta}{\gamma}\right)^{\tau/\sigma} + \int_{(\gamma/4\delta)^{1/\sigma}}^{\infty} y^{-\tau}\,\mathrm{d}y.$$

According to lemma 5 we obtain

$$|[-\delta\omega, \delta\omega] \setminus \Lambda| \leqslant \frac{\omega\gamma}{2}\,(2\delta)^{\rho}\left(\left(\frac{4\delta}{\gamma}\right)^{\tau/\sigma} + \frac{1}{\tau-1}\left(\frac{4\delta}{\gamma}\right)^{(\tau-1)/\sigma}\right)$$

and so, assuming $\tau > 1 + \sigma(1-\rho)$,

$$\lim_{\delta\downarrow 0} \frac{1}{2\delta\omega}\,|[-\delta\omega, \delta\omega] \setminus \Lambda| = 0. \qquad \square$$

Suppose that the sequence E obeys the gap condition (2) with $\alpha > 0$. A possible choice of the constants σ, τ and ρ which suits the assumption of lemma 6 is

$$\tau = 1+\alpha \qquad 1 < \sigma < 1+\alpha \qquad \text{and} \qquad \rho = 1/\sigma.$$

In our model we have effectively $\alpha = 1$ and so we choose

$$\tau = 2 \qquad 1 < \sigma < 2 \qquad \text{and} \qquad \rho = 1/\sigma \in\,]1/2, 1[. \tag{16}$$

Let us now derive some consequences of the above diophantine estimates in combination with the gap condition (2). Suppose once again that the spectrum of H is pure point and equals $\{E(k)\}_{k\in\mathbb{Z}_+}$, $E(0) = 0$, and that E obeys the gap condition (2). It is quite useful to observe that another inequality follows straightforwardly from (2):

$$|E(j) - E(k)| \geqslant \frac{C_E}{\alpha+1}\,|j-k| \max\{j^{\alpha}, k^{\alpha}\} \qquad \forall\, j, k \in \mathbb{Z}_+. \tag{17}$$

We shall denote by P_n, $n \in \mathbb{Z}\times\mathbb{Z}_+$ (or $\mathbb{Z}\times\mathbb{Z}$ in our model), the eigenprojectors of K corresponding to the eigenvalues $F(n)$; we have $P \equiv P_{\bar{0}}$ with $F(\bar{0}) = 0$. We set also $Q_n := 1 - P_n$.

Another important observation coming from the gap condition is that those eigenstates P_n which can potentially contribute by small denominators are distributed rather rarely in the half-plane $n_2 \geqslant 0$. Let $\mathcal{S}$ designate the set of 'critical' indices defined by

$$n \in \mathcal{S} \text{ iff } \ F(n) \in\,]-\omega/2, \omega/2\,] \setminus \{0\}. \tag{18}$$

Clearly, for each $n_2 \in \mathbb{N}$ there exists exactly one $n_1 \in \mathbb{Z}$ (necessarily $n_1 \leqslant 0$) such that $n \in \mathcal{S}$; $(n_1, 0) \notin \mathcal{S}$ for all $n_1 \neq 0$, and we treat $n = \bar{0}$ separately, since it corresponds to the eigenstate P to be perturbed. Furthermore, if $m, n \in \mathcal{S}$ and $m_2 \leqslant n_2$ then $|m_1| \leqslant |n_1|$. Roughly speaking, the indices from the set $\mathcal{S}$ are situated close to the curve $n_1 = -E(n_2)/\omega$. We set $P_{\mathcal{S}} := \sum_{n\in\mathcal{S}} P_n$, $Q_{\mathcal{S}} := Q - P_{\mathcal{S}}$. Evidently, $\|\Gamma_0 Q_{\mathcal{S}}\| \leqslant 2/\omega$.

Let us introduce a function L defined on $\mathcal{S}$:

$$L(n) := \min\{|n_2|,\ d(n)\} \tag{19}$$

where

$$d(n) := \operatorname{dist}(n_1, \operatorname{pr}_1(\mathcal{S}\setminus\{n\})) = \min_{n'\in\mathcal{S},\ |n'_2-n_2|=1} |n'_1 - n_1| \leqslant \operatorname{dist}(n_1, \operatorname{pr}_1(\mathcal{S}) \setminus \{n_1\})$$

and pr_1 is the projection onto the first coordinate axis.

Lemma 7. Assume that the function $\tilde{\psi}$ occurring in the definition (14) of the set Λ satisfies

$$\sup_{k\in\mathbb{N}} \; k^{-\min\{1,\alpha\}}\,|\log\tilde{\psi}(k)| < \infty.$$

Then there exists a constant $C_1 > 1$ such that

$$|F(n)-\lambda| \geqslant (2|\lambda|/\omega)^{\rho}\, C_1^{-L(n)} \qquad \forall n\in\mathcal{S} \quad \forall\lambda\in\Lambda.$$

Proof. It is sufficient to find C_1 so that

$$\omega\gamma\,\tilde{\psi}(n_2) \geqslant \max\{C_1^{-n_2},\; C_1^{-d(n)}\}$$

holds for all $n\in\mathcal{S}$. Observe that for any couple $m,n\in\mathcal{S}$, $m\neq n$, we have $m_2\neq n_2$ and

$$\omega|n_1-m_1| \geqslant |E(n_2)-E(m_2)| - |F(n)-\lambda| - |F(m)-\lambda|$$

and consequently, by virtue of (17) and the definition (18) of $\mathcal{S}$

$$d(n) \geqslant (C_E/(\alpha+1))\,|n_2|^{\alpha} - \omega. \tag{20}$$

The rest of the proof is self-evident. □

We are going to verify one more estimate related to the function $L(n)$ defined in (19). To this end we shall need the following lemma.

Lemma 8. Let $\Delta_0, \Delta_1, \ldots, \Delta_\ell$ be a family of positive numbers. Then it holds that

$$\left|\frac{1}{\Delta_1+\Delta_2+\cdots+\Delta_\ell} - \frac{1}{\ell\Delta_0}\right| \leqslant \max_{1\leqslant k\leqslant \ell}\left|\frac{1}{\Delta_k}-\frac{1}{\Delta_{k-1}}\right|.$$

Proof. The proof follows immediately from the identity

$$\begin{aligned}\frac{1}{\Delta_1+\Delta_2+\cdots+\Delta_\ell} &- \frac{1}{\ell\Delta_0}\\ &= \frac{1}{\ell}\left[\left(\frac{1}{\Delta_1}-\frac{1}{\Delta_0}\right)(\Delta_1+\cdots+\Delta_\ell) + \left(\frac{1}{\Delta_2}-\frac{1}{\Delta_1}\right)(\Delta_2+\cdots+\Delta_\ell)\right.\\ &\quad \left. +\cdots+\left(\frac{1}{\Delta_\ell}-\frac{1}{\Delta_{\ell-1}}\right)\Delta_\ell\right]\frac{1}{\Delta_1+\cdots+\Delta_\ell}.\end{aligned}$$

□

Let us define

$$\Delta E(k) := E(k+1)-E(k), \;\; k\in\mathbb{Z}_+$$

and suppose that E still satisfies the gap condition (2), $E(0)=0$. Concerning the function $\tilde{\psi}$ we assume that it is decreasing and

$$\sup_{k\in\mathbb{N}} \tilde{\psi}(k)/\tilde{\psi}(2k) =: C_\psi < \infty. \tag{21}$$

The following lemma contains a condition relating the sequences ΔE and $\tilde{\psi}$.

Lemma 9. Assume that

$$\sup_{k\in\mathbb{Z}_+}\frac{1}{\tilde{\psi}(k)}\left|\frac{1}{\Delta E(k+1)}-\frac{1}{\Delta E(k)}\right| =: C_\Delta < \infty. \tag{22}$$

Then there exists a constant $C_2 > 0$ such that for each $n\in\mathcal{S}$ verifying

$$\min\{\Delta E(n_2),\; \Delta E(n_2-1)\} \geqslant 4\omega \tag{23}$$

and for all $m \in \mathbb{Z} \times \mathbb{N}$, $m \neq n$, from the neighbourhood

$$2\max\{|n_1 - m_1|, |n_2 - m_2|\} \leqslant L(n) \tag{24}$$

and for all $\lambda \in \Lambda \cap [-\omega/3, \omega/3]$, it holds true that

$$\left| \frac{1}{F(m) - \lambda} + \frac{1}{F(m') - \lambda} \right| \leqslant C_2\, (2|\lambda|/\omega)^{-\rho}\, |F(n) - \lambda|$$

where $m' = 2n - m$.

Proof. The assumptions have some obvious consequences. First, (23) implies that if $m \in \mathcal{S}$ and $m_2 = n_2 \pm 1$ then $|m_1 - n_1| \geqslant 3$. Furthermore, the condition

$$2|n_1 - m_1| \leqslant \mathrm{dist}(n_1, \mathrm{pr}_1(\mathcal{S} \setminus \{n\})) \qquad \text{and} \qquad m \neq n$$

implies that $m \notin \mathcal{S}$. Thus one finds that

$$|F(m) - \lambda| \geqslant \left(\tfrac{1}{2} - \tfrac{1}{3}\right)\omega = \tfrac{1}{6}\,\omega.$$

Obviously, equation (24) also implies that $n_2/2 \leqslant m_2 \leqslant 3n_2/2$. Second, we have

$$|F(m) - \lambda| \geqslant |E(m_2) - E(n_2)|/6. \tag{25}$$

Indeed, if $m_2 \neq n_2$ then

$$|F(m) - \lambda| \geqslant |E(m_2) - E(n_2)| \left(1 - \frac{\omega|m_1 - n_1| + |F(n)| + |\lambda|}{|E(m_2) - E(n_2)|}\right).$$

Let $n' \in \mathcal{S}$ be such that $|n_2' - n_2| = 1$ and $\mathrm{sgn}(n_2' - n_2) = \mathrm{sgn}(m_2 - n_2)$. Then $\mathrm{dist}(n_1, \mathrm{pr}_1(\mathcal{S} \setminus \{n\})) \leqslant |n_1 - n_1'|$ and, owing to (24),

$$\begin{aligned} 2\omega|n_1 - m_1| &\leqslant \omega|n_1 - n_1'| = |E(n_2') - E(n_2) + F(n) - F(n')| \\ &\leqslant |E(m_2) - E(n_2)| + \left(\frac{\omega}{2} + \frac{\omega}{2}\right). \end{aligned}$$

Note that ($m_2 \neq n_2$)

$$|E(m_2) - E(n_2)| \geqslant \min\{\Delta E(n_2), \Delta E(n_2 - 1)\} \geqslant 4\omega.$$

Altogether this means that

$$\frac{\omega|m_1 - n_1| + |F(n)| + |\lambda|}{|E(m_2) - E(n_2)|} \leqslant \frac{1}{2} + \left(\frac{\omega}{2} + \frac{\omega}{2} + \frac{\omega}{3}\right)\frac{1}{4\omega} = \frac{5}{6}$$

and (25) follows. All the above estimates are also valid for m'. Now write

$$\frac{1}{F(m) - \lambda} + \frac{1}{F(m') - \lambda} = \frac{2(F(n) - \lambda) + E(m_2) + E(m_2') - 2E(n_2)}{(F(m) - \lambda)(F(m') - \lambda)}.$$

Now to finish the proof, it suffices to study the case $m_2 - n_2 = n_2 - m_2' \neq 0$. From (25) one finds that

$$\begin{aligned} 6^{-2} \left| \frac{E(m_2) + E(m_2') - 2E(n_2)}{(F(m) - \lambda)(F(m') - \lambda)} \right| &\leqslant \left| \frac{1}{E(m_2) - E(n_2)} + \frac{1}{E(m_2') - E(n_2)} \right| \\ &\leqslant \left| \frac{1}{E(m_2) - E(n_2)} - \frac{1}{(m_2 - n_2)\Delta E(n_2)} \right| \\ &\quad + \left| \frac{1}{E(m_2') - E(n_2)} - \frac{1}{(m_2' - n_2)\Delta E(n_2)} \right|. \end{aligned} \tag{26}$$

Combining lemma 8, the monotonic behaviour of $\tilde{\psi}$, and assumption (22) we obtain

$$\left| \frac{1}{E(j+\ell)-E(j)} - \frac{1}{\ell\,\Delta E(j)} \right| \leqslant C_\Delta\,\tilde{\psi}(j)$$

$$\left| \frac{1}{E(j)-E(j-\ell)} - \frac{1}{\ell\,\Delta E(j)} \right| \leqslant C_\Delta\,\tilde{\psi}(j-\ell).$$

Thus we can estimate the right-hand side of (26) from the above by (cf equation (21))

$$\begin{aligned} 2C_\Delta\,\tilde{\psi}(\min\{m_2, m_2'\}) &\leqslant 2C_\Delta C_\psi\,\tilde{\psi}(2\min\{m_2, m_2'\}) \leqslant 2C_\Delta C_\psi\,\tilde{\psi}(n_2) \\ &\leqslant (2C_\Delta C_\psi/\omega\gamma)\,(2|\lambda|/\omega)^{-\rho}\,|F(n)-\lambda|. \end{aligned}$$

This completes the proof. □

Finally note that, with the choice of $\tilde{\psi}$ (15) and for $E(k) = k^2$, the assumptions of both lemmas 7 and 8 are satisfied. Thus these two lemmas are applicable to our example provided the choices (15) and (16) have been made.

4. Solution of the fixed-point equation

We wish to justify the power series

$$g(\beta,\lambda) = \sum_{k=0}^{\infty} (-\beta)^{k+1}\,(\Gamma_\lambda \hat{V})^k\,\Gamma_\lambda Q V f \tag{27}$$

as a solution to the vector equation (6). We start from an estimate whose proof relies heavily on the very special features of our model. This does not concern the spectrum of H (the gap condition (2) would be sufficient) but what is really special is the form of the potential (12). For each $m \in \mathbb{Z}^2$ there exist exactly four indices $n \in \mathbb{Z}^2$ such that $V_{mn} \neq 0$. This fact makes it possible to use some elementary combinatorics in order to treat the summands in (27). The heart of the proof is a sort of compensation based on lemma 9. This method of compensations was inspired by the pioneering work of Eliasson (1988).

Recall the definition of the lattice $\mathbb{L}$ (section 2) and denote by $\mathcal{P}(N) \subset (\mathbb{Z}^2)^{N+1}$ the set of (unclosed) paths in $\mathbb{L}$ of length N with the initial vertex $\bar{0}$: $(\bar{\iota}(0), \bar{\iota}(1), \ldots, \bar{\iota}(N)) \in \mathcal{P}(N)$ iff $\bar{\iota}(0) = \bar{0}$, $\bar{\iota}(j) \neq \bar{0}$ for $1 \leqslant j \leqslant N$, and $\bar{\iota}(j) - \bar{\iota}(j-1) \in \{\pm(1,1), \pm(1,-1)\}$ for $1 \leqslant j \leqslant N$. Clearly, $|\mathcal{P}(N)| \leqslant 4^N$. For $M \in \mathbb{N}$ one can write

$$(\Gamma_\lambda \hat{V})^{M-1}\Gamma_\lambda Q V P = \sum_{\bar{\iota}\in\mathcal{P}(M)} \left(\prod_{j=1}^{M} \frac{1}{F(\bar{\iota}(j))-\lambda} \right) P_{\bar{\iota}(M)}. \tag{28}$$

Lemma 10. In the case of the model (1) and assuming that the choices (15) and (16) have been made, there exists a constant $\hat{C} > 0$ such that

$$\|\Gamma_\lambda Q V f\| \leqslant \hat{C} \qquad \|(\Gamma_\lambda \hat{V})^{M-1}\Gamma_\lambda Q V f\| \leqslant \left(\frac{2|\lambda|}{\omega}\right)^{\rho} \left(\left(\frac{2|\lambda|}{\omega}\right)^{-\rho/2} \hat{C} \right)^{M}$$

holds true for $\forall\, M \in \mathbb{N},\ M \geqslant 2$, and $\forall\, \lambda \in \Lambda \cap [-\omega/3, \omega/3],\ \lambda \neq 0$.

Remark. Note the type of the estimate: we are able to estimate the vector $(\Gamma_\lambda \hat{V})^{M-1}\Gamma_\lambda Q V f$ but not directly the operator $(\Gamma_\lambda \hat{V})^M$.

Proof. We start by restricting the set $\mathcal{S}$ of critical indices to a subset $\mathcal{S}' = \{n \in \mathcal{S};\ |n_2| > b\}$. The bound $b \in \mathbb{N}$ is required to obey the conditions:

- $b \geqslant 3$,
- $4\omega \leqslant \min\{\Delta(k), \Delta(k-1)\}$ for $\forall\, k > b$,
- $L(n) \geqslant 2$ for $\forall\, n \in \mathcal{S},\ |n_2| > b$.

The second requirement is dictated by the assumption (23) of lemma 9 and the third one is possible since from the estimate (20) it follows that

$$\lim_{n \in \mathcal{S},\ |n_2| \to \infty} L(n) = +\infty.$$

Clearly, since $|F(n) - \lambda| \geqslant \omega/6$ for $n \notin \mathcal{S},\ |\lambda| \leqslant \omega/3$, there exists a constant $C_3 > 0$ such that

$$|F(n) - \lambda| \geqslant C_3 \qquad \text{for } \forall\, n \notin \mathcal{S}' \cup \{\bar{0}\},\ \forall\, \lambda \in \Lambda \cap [\,-\omega/3, \omega/3\,].$$

Without loss of generality we can restrict ourselves to $M \geqslant 2$. For each $\bar{\iota} \in \mathcal{P}(M)$ the vertices from $\mathcal{S}'$ split the path into segments. Consider such a segment of length ℓ, $(\bar{\iota}(j), \bar{\iota}(j+1), \ldots, \bar{\iota}(j+\ell))$, with $\bar{\iota}(j+\ell) \in \mathcal{S}'$, and also $\bar{\iota}(j) \in \mathcal{S}'$ provided $j \neq 0$, and $\bar{\iota}(j+s) \notin \mathcal{S}'$ for $1 \leqslant s \leqslant \ell - 1$. However, in order not to count it twice, we do not relate the contribution from the vertex $\bar{\iota}(j)$ with the segment.

We distinguish two cases. If $\ell \geqslant L(\bar{\iota}(j+\ell))$ then lemma 7 implies

$$\left| \prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{\iota}(s)) - \lambda} \right| \leqslant \left(\frac{1}{C_3} \right)^{\ell-1} \left(\frac{2|\lambda|}{\omega} \right)^{-\rho} C_1^{\ell}. \tag{29}$$

Consider now the case $\ell < L(\bar{\iota}(j+\ell))$. The possibility $j = 0$ is excluded since this would imply $\ell < |\bar{\iota}(\ell)_2| \leqslant \ell$. Thus $\bar{\iota}(j), \bar{\iota}(j+\ell) \in \mathcal{S}'$ and necessarily $\bar{\iota}(j) = \bar{\iota}(j+\ell)$ as follows from

$$|\bar{\iota}(j+\ell)_1 - \bar{\iota}(j)_1| \leqslant \ell < \operatorname{dist}(\bar{\iota}(j+\ell)_1,\ \operatorname{pr}_1(\mathcal{S}) \setminus \{\bar{\iota}(j+\ell)_1\}).$$

Consequently, ℓ is even. We shall call a segment of this type short loop. To any short loop there exists an opposite short loop $(\bar{\iota}'(j), \bar{\iota}'(j+1), \ldots, \bar{\iota}'(j+\ell) = \bar{\iota}'(j))$ defined by $\bar{\iota}'(s) := 2\bar{\iota}(j) - \bar{\iota}(s),\ j \leqslant s \leqslant j + \ell$; hence the base point is the same, $\bar{\iota}'(j) = \bar{\iota}(j)$. Now we are approaching the compensation step. The contribution of two opposite short loops equals

$$\begin{aligned} &\prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{\iota}(s)) - \lambda} + \prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{\iota}'(s)) - \lambda} \\ &\qquad = \frac{1}{F(\bar{\iota}(j)) - \lambda} \left(\prod_{s=j+1}^{j+\ell-1} \frac{1}{F(\bar{\iota}(s)) - \lambda} - \prod_{s=j+1}^{j+\ell-1} \frac{1}{-F(\bar{\iota}'(s)) + \lambda} \right). \end{aligned} \tag{30}$$

In order to estimate the difference of products on the right-hand side of (30) one can use the identity

$$u_1 \cdots u_N - v_1 \cdots v_N = \sum_{s=1}^{N} u_1 \cdots u_{s-1} (u_s - v_s) v_{s+1} \cdots v_N \tag{31}$$

and lemma 9. In this way one arrives at

$$|\text{expression (30)}| \leqslant (\ell - 1) \left(\frac{1}{C_3} \right)^{\ell-2} C_2 \left(\frac{2|\lambda|}{\omega} \right)^{-\rho} \leqslant C_2 C_3^2 \left(\frac{2|\lambda|}{\omega} \right)^{-\rho} \left(\frac{2}{C_3} \right)^{\ell}. \tag{32}$$

In order to treat this type of compensation systematically let us split $\mathcal{P}(M)$ into equivalence classes. Two paths are equivalent if and only if one is obtained from the other by replacing several short loops by their opposites. Thus a path containing s short loops belongs to a class with 2^s elements. Schematically one can write

$$\sum_{\text{all paths}} \prod_{\text{all segments}} = \sum_{\text{equivalence classes}} \prod_{\text{pairs of short loops}} \times \prod_{\text{other segments}}$$

For a path $\bar{\iota} \in \mathcal{P}(M)$ denote by $N = N(\bar{\iota})$ the number of vertices belonging to $\mathcal{S}'$. Obviously, $N(\bar{\iota})$ is constant on every equivalence class. Relying on the estimates (29) and (32) one concludes readily that there exists a constant $\hat{C} > 0$ such that

$$\left| \sum_{\text{equivalence class}} \prod_{j=1}^{M} \frac{1}{F(\bar{\iota}(j)) - \lambda} \right| \leqslant \left(\frac{2|\lambda|}{\omega} \right)^{-\rho N} \left(\frac{\hat{C}}{4} \right)^{M}.$$

Since $b \geqslant 3$ we have $\bar{\iota}(1), \bar{\iota}(2), \bar{\iota}(3) \notin \mathcal{S}'$ and consequently, as $L(n) \geqslant 2$ for all $n \in \mathcal{S}'$,

$$2N(\bar{\iota}) \leqslant M - 2.$$

To complete the proof it is sufficient to estimate the number of equivalence classes from the above simply by $|\mathcal{P}(M)| \leqslant 4^M$ (cf equation (28)). □

With the estimate given in lemma 10, it is quite straightforward to derive the following existence (but not uniqueness) result.

Lemma 11. Under the same assumptions as in lemma 10, the series (27) converges to a solution $g(\beta, \lambda)$ of the equation (6) provided (β, λ) belongs to the domain

$$\lambda \in \Lambda \cap [-\omega/3, \omega/3] \qquad |\beta| \leqslant (2|\lambda|/\omega)^{\rho/2}/2\hat{C}. \tag{33}$$

For each $\lambda \in \Lambda \cap [-\omega/3, \omega/3]$, $\lambda \neq 0$, the vector-valued function $g(\beta, \lambda)$ is analytic in β on the corresponding neighbourhood of 0 and

$$\|g(\beta, \lambda) + \beta \Gamma_\lambda Q V f\| \leqslant 2\hat{C}^2 \beta^2. \tag{34}$$

Now we can give a precise meaning to the right-hand side of the fixed-point equation (8). For (β, λ) from the domain (33)

$$\begin{aligned} G(\beta, \lambda) &:= \beta \, \langle QVf, \; g(\beta, \lambda) \rangle = \sum_{k=1}^{\infty} \beta^{2k} \, G_{2k}(\lambda) \\ G_{2k}(\lambda) &:= -\langle QVf, \; (\Gamma_\lambda \hat{V})^{2k-2} \Gamma_\lambda QVf \rangle. \end{aligned} \tag{35}$$

In our particular example we have $G_{2k+1}(\lambda) = 0$ for $k \geqslant 1$ but generally this need not be the case. As a consequence of lemma 10 we obtain

$$|G_{2k}(\lambda)| \leqslant \|V\| \left(\frac{2|\lambda|}{\omega} \right)^{\rho} \left(\left(\frac{2|\lambda|}{\omega} \right)^{-\rho/2} \hat{C} \right)^{2k-1}. \tag{36}$$

For our model in particular $(E(1) = 1)$

$$G_2(\lambda) = -\langle QVf, \; \Gamma_\lambda QVf \rangle = \frac{4(E(1) - \lambda)}{\omega^2 - (E(1) - \lambda)^2}$$

and $G_2(0) \neq 0$.

We shall impose a stricter bound on λ, $|\lambda| \leqslant \lambda_\star$, where $0 < \lambda_\star \leqslant \omega/3$, and we require $\lambda_\star$ to be sufficiently small so that

- $|G_2(\lambda) - G_2(0)| \leqslant |G_2(0)|/2$,
- $(2\lambda_\star/\omega)^{1-\rho} \leqslant |G_2(0)|/(8\omega\hat{C}^2)$,

- $\lambda_\star^{1/2} \leqslant |G_2(0)|^{3/2}/(16\,\|V\|\,\hat{C}^2)$,
- $(2\lambda_\star/\omega)^{\rho/2} \leqslant |G_2(0)|/(2\,\|V\|\,\hat{C})$.

Recall that $\frac{1}{2} < \rho < 1$ (cf equation (16)). Set

$$B(\lambda) := 2\,(|\lambda|/|G_2(0)|)^{1/2}.$$

The first requirement implies $|G_2(\lambda)| \geqslant |G_2(0)|/2$ and $\operatorname{sgn} G_2(\lambda) = \operatorname{sgn} G_2(0)$. Owing to the second requirement we have

$$|\lambda| \leqslant \lambda_\star \Longrightarrow B(\lambda) \leqslant (2|\lambda|/\omega)^{\rho/2}/2\hat{C}$$

and so the conditions $\lambda \in \Lambda \cap [-\lambda_\star, \lambda_\star]$, $|\beta| \leqslant B(\lambda)$ determine a subdomain of (33). From the third requirement it follows that

$$|\lambda| \leqslant \lambda_\star \Longrightarrow 2\|V\|\,\hat{C}^2 B(\lambda)^3 \leqslant |\lambda|. \tag{37}$$

Finally, a routine calculation based on the definition (35) of G, the estimate (36), and the fourth requirement yields the inequality

$$|\partial_\beta G(\beta,\lambda) - 2\beta G_2(\lambda)| < |\beta|\,|G_2(0)| \leqslant 2|\beta|\,|G_2(\lambda)| \tag{38}$$

valid for $0 < |\lambda| \leqslant \lambda_\star,\ 0 < |\beta| \leqslant (2|\lambda|/\omega)^{\rho/2}/2\hat{C}$. Consequently

$$\operatorname{sgn} \partial_\beta G(\beta,\lambda) = \operatorname{sgn} \beta G_2(\lambda) = \operatorname{sgn} \beta G_2(0). \tag{39}$$

Lemma 12. Under the same assumptions as in lemma 10, for each $\lambda \in \Lambda \cap [-\lambda_\star, \lambda_\star]$, $\operatorname{sgn} \lambda = \operatorname{sgn} G_2(0)$, there exist exactly two solutions $\beta_\pm(\lambda)$ to the equation $\lambda = G(\beta,\lambda)$ in the interval $[-B(\lambda), B(\lambda)]$, and there is no solution for $\operatorname{sgn}\lambda = -\operatorname{sgn} G_2(0)$. The two solutions are non-zero, differ in sign, and we choose the convention

$$-B(\lambda) \leqslant \beta_-(\lambda) < 0 < \beta_+(\lambda) \leqslant B(\lambda).$$

Then λ is an eigenvalue of the operators $K + \beta_\pm(\lambda)\,V$.

Remark. Since, in the case of our model, $G(\beta,\lambda)$ is even in β we have consequently $\beta_-(\lambda) = -\beta_+(\lambda)$. But, of course, this is not a general feature.

Proof. Obviously, $G(0,\lambda) = 0$. Let us show that $|G(\pm B(\lambda),\lambda)| \geqslant |\lambda|$. From equation (34) we obtain

$$|G(\beta,\lambda) - \beta^2 G_2(\lambda)| = |\beta\,\langle QVf,\ g(\beta,\lambda) + \beta\Gamma_\lambda QVf\rangle| \leqslant 2\|V\|\,\hat{C}^2|\beta|^3$$

and, owing to (37),

$$|G(\pm B(\lambda),\lambda) - B(\lambda)^2\,G_2(\lambda)| \leqslant |\lambda|.$$

On the other hand,

$$|B(\lambda)^2\,G_2(\lambda)| \geqslant 4\frac{|\lambda|}{|G_2(0)|}\,\frac{1}{2}\,|G_2(0)| = 2|\lambda|.$$

In this way we have also verified that

$$\operatorname{sgn} G(\pm B(\lambda),\lambda) = \operatorname{sgn} G_2(\lambda) = \operatorname{sgn} G_2(0).$$

Now existence follows from the fact that the function $G(\beta,\lambda)$ is continuous (even analytic) in β. The uniqueness is a consequence of the monotonic behaviour (cf equation (39)). □

5. Properties of the function $\boldsymbol{\lambda}(\beta)$

We intend to invert the functions $\beta_+(\lambda)$ and $\beta_-(\lambda)$ in order to obtain the desired function $\lambda(\beta)$ defined respectively on sets I_+ and I_-, with $I_\pm \subset \mathbb{R}_\pm$, and we naturally set $\lambda(0) = 0$. Thus the total domain for $\lambda(\beta)$ is $I = I_- \cup \{0\} \cup I_+$. $\lambda(\beta)$ is positive (negative), except of $\lambda(0) = 0$, if $G_2(0)$ is positive (negative). The existence of the inverted function follows from the monotonic behaviour of the original functions $\beta_\pm(\lambda)$.

We shall need the following lemma.

Lemma 13. The function $G(\beta, \lambda)$ defined in (35) fulfills the equality

$$G(\beta, \lambda_2) - G(\beta, \lambda_1) = -(\lambda_2 - \lambda_1)\, \langle g(\beta, \lambda_2),\ g(\beta, \lambda_1)\rangle$$

for all

$$\lambda_1, \lambda_2 \in \Lambda \cap [\,-\omega/3, \omega/3\,],\ \ |\beta| \leqslant (2\min\{|\lambda_1|, |\lambda_2|\}/\omega)^{\rho/2}/2\hat{C}. \tag{40}$$

Proof. Note that $\Gamma_{\lambda_2} - \Gamma_{\lambda_1} = (\lambda_2 - \lambda_1)\, \Gamma_{\lambda_2}\Gamma_{\lambda_1}$ on $\mathcal{D}(\Gamma_{\lambda_1}) \cap \mathcal{D}(\Gamma_{\lambda_2})$ and consequently, using (31),

$$\langle QVf,\ (\Gamma_{\lambda_2}\hat{V})^k \Gamma_{\lambda_2} QVf - (\Gamma_{\lambda_1}\hat{V})^k \Gamma_{\lambda_1} QVf\rangle$$

$$= (\lambda_2 - \lambda_1) \sum_{j=0}^{k} \langle (\Gamma_{\lambda_2}\hat{V})^j \Gamma_{\lambda_2} QVf,\ (\Gamma_{\lambda_1}\hat{V})^{k-j} \Gamma_{\lambda_1} QVf\rangle.$$

Now the identity can be verified easily with the aid of (27). □

From equation (34) one deduces that $\langle g(\beta, \lambda_2),\ g(\beta, \lambda_1)\rangle > 0$ whenever $|\lambda_1|, |\lambda_2|$ are sufficiently small and $|\beta|$ obeys (40). Thus we find that $G(\beta, \lambda)$ is strictly decreasing in λ for every β fixed. The same is true for the function $\Phi(\beta, \lambda) := G(\beta, \lambda) - \lambda$.

This is an elementary exercise to verify that the functions $\beta_\pm(\lambda)$ are strictly monotonic provided one uses the equality $\Phi(\beta_\pm(\lambda), \lambda) = 0$ and the fact that $\Phi(\beta, \lambda)$ is monotonic in β (c.f. (39)) and strictly monotonic in λ. We can formulate our conclusion as follows.

Lemma 14. There exists a bound $\lambda_{\star\star}$, $0 < \lambda_{\star\star} \leqslant \lambda_\star$, and a function $\lambda(\beta)$ defined on $I \subset \mathbb{R}$ such that $0 \in I$ and $\lambda(0) = 0$, $\beta_\pm(\lambda(\beta)) = \beta$ for $\forall\beta \in I \cap \mathbb{R}_\pm$, and the range of both $\lambda(\beta)|I \cap \mathbb{R}_+$ and $\lambda(\beta)|I \cap \mathbb{R}_-$ equals either $\Lambda \cap [\,0, \lambda_{\star\star}\,]$ or $\Lambda \cap [\,-\lambda_{\star\star}, 0\,]$ depending on whether $G_2(0)$ is positive or negative. For $\forall\beta \in I$, $\lambda(\beta)$ is an eigenvalue of the operator $K + \beta V$.

That one has to abandon some values of the coupling constant β and determine the perturbed eigenvalue as a function $\lambda(\beta)$ defined on a domain I possessing 'holes' seems to be a typical feature of the perturbation theory of dense point spectra. To treat functions of this type one can refer to the celebrated Whitney Extension Theorem (see Stein 1970). In fact, its proof in the one-dimensional case is rather elementary. We shall need the following very particular version.

Lemma 15. Let χ be a real function defined on a closed subset $Y \subset \mathbb{R}$, χ being monotonic, and suppose that there exist two constants $0 < A \leqslant B$ such that

$$A|y_1 - y_2| \leqslant |\chi(y_1) - \chi(y_2)| \leqslant B|y_1 - y_2| \qquad \text{for all}\ \ y_1, y_2 \in Y.$$

Then there exists an extension $\tilde{\chi}$ defined on $\mathbb{R}$, $\tilde{\chi}|Y = \chi$, and $\tilde{\chi}$ is again monotonic and obeys the same inequalities but this time on the whole line $\mathbb{R}$,

$$A|y_1 - y_2| \leqslant |\tilde{\chi}(y_1) - \tilde{\chi}(y_2)| \leqslant B|y_1 - y_2| \qquad \text{for all}\ \ y_1, y_2 \in \mathbb{R}.$$

Proof. The complement of Y is an open subset of $\mathbb{R}$ and hence at most countable disjoint union of open intervals. One defines the function $\tilde{\chi}$ linearly on these intervals requiring it to be continuous. Provided the interval in question is half-infinite then $\tilde{\chi}$ is defined again linearly with the slope lying between A and B. The inequalities for $\tilde{\chi}$ defined this way are easy to verify; for the left one we require that χ be monotonic. □

We wish to show that 0 is a point of density for the set I. We already know that this is true for the set Λ (lemma 6). The intermediate step is given by

Lemma 16. Assume that a real function $\varphi(x)$, defined on a set $X \subset [\,0, +\infty[$, is strictly increasing, $\varphi(0) = 0$ ($\Rightarrow 0 \in X$), and the set $Y = \varphi(X)$ is closed. Moreover, suppose that there exist two constants $0 < A \leqslant B$ such that

$$A|x_1^2 - x_2^2| \leqslant |\varphi(x_1) - \varphi(x_2)| \leqslant B|x_1^2 - x_2^2| \qquad \text{for all } \; x_1, x_2 \in X. \tag{41}$$

Then it holds that

$$\lim_{\eta\downarrow 0} |Y \cap [\,0, \eta\,]|/\eta = 1 \implies \lim_{\delta\downarrow 0} |X \cap [\,0, \delta\,]|/\delta = 1. \tag{42}$$

Proof. Apply lemma 14 to the function $\chi(y) = (\varphi^{-1}(y))^2$ (the corresponding constants are $0 < 1/B \leqslant 1/A$). The extension $\tilde{\chi}$ is again strictly increasing, $\tilde{\chi}(y) > 0$ for $y > 0$, and $\tilde{\chi}(\mathbb{R}_+) = \mathbb{R}_+$. Define $\tilde{\varphi}$ on $\mathbb{R}_+$ by $\tilde{\varphi}(x) = y$ iff $x^2 = \tilde{\chi}(y)$, i.e. $\tilde{\varphi}$ is the inverse of $(\tilde{\chi}|\mathbb{R}_+)^{1/2}$. Clearly, the function $\tilde{\varphi}$ is an extension of φ, $\tilde{\varphi}|X = \varphi$, it is again strictly increasing, and the inequalities (41) hold for $\tilde{\varphi}$ on the whole positive half-line. Consequently, $\tilde{\varphi}$ is absolutely continuous on every bounded interval, $\tilde{\varphi}'$ exists almost everywhere, and it holds that

$$\tilde{\varphi}(x) \leqslant B\, x^2 \qquad \text{and} \qquad 2Ax \leqslant \tilde{\varphi}'(x) \qquad \text{for (almost) all } \; x \geqslant 0.$$

Denote by X^{c} and Y^{c} the complements of X and Y in $[\,0, +\infty[$, respectively. The implication (42) is equivalent to

$$\lim_{\eta\downarrow 0} |Y^{\mathrm{c}} \cap [\,0, \eta\,]|/\eta = 0 \implies \lim_{\delta\downarrow 0} |X^{\mathrm{c}} \cap [\,0, \delta\,]|/\delta = 0. \tag{43}$$

Choose p, $1 < p < 2$, and let q be the adjoint exponent, $p^{-1} + q^{-1} = 1$. We shall verify the inequality

$$\delta^{-1}\,|X^{\mathrm{c}} \cap [\,0, \delta\,]| \leqslant \frac{B}{2A}\,\left(1 - \frac{p}{2}\right)^{-1/p}\,\left(\tilde{\varphi}(\delta)^{-1}\,|Y^{\mathrm{c}} \cap [\,0, \tilde{\varphi}(\delta)\,]|\right)^{1/q}. \tag{44}$$

It is clear that (43) is a consequence of (44). We have

$$|X^{\mathrm{c}} \cap [\,0, \delta\,]| = \int_{Y^{\mathrm{c}}\cap[\,0,\tilde{\varphi}(\delta)\,]} \frac{\mathrm{d}y}{\tilde{\varphi}'(\tilde{\varphi}^{-1}(y))} \leqslant \frac{\sqrt{B}}{2A} \int_{Y^{\mathrm{c}}\cap[\,0,\tilde{\varphi}(\delta)\,]} y^{-1/2}\,\mathrm{d}y$$

since $\tilde{\varphi}'(\tilde{\varphi}^{-1}(y)) \geqslant 2A\,\tilde{\varphi}^{-1}(y) \geqslant 2A\,(y/B)^{1/2}$. The Hölder inequality then gives

$$\begin{aligned}\int_{Y^{\mathrm{c}}\cap[\,0,\tilde{\varphi}(\delta)\,]} y^{-1/2}\,\mathrm{d}y &\leqslant \left(\int_0^{\tilde{\varphi}(\delta)} y^{-p/2}\,\mathrm{d}y\right)^{1/p} \left(\int_{Y^{\mathrm{c}}\cap[\,0,\tilde{\varphi}(\delta)\,]} \mathrm{d}y\right)^{1/q} \\ &\leqslant \left(1 - \frac{p}{2}\right)^{-1/p} \sqrt{B}\delta\,\left(\tilde{\varphi}(\delta)^{-1}\,|Y^{\mathrm{c}} \cap [\,0, \tilde{\varphi}(\delta)\,]|\right)^{1/q}\end{aligned}$$

and (44) follows immediately. □

Observe that the property (2) given in proposition 1 is equivalent to

$$\lim_{\delta\downarrow 0} |I \cap [\,0, \delta\,]|/\delta = 1 \qquad \text{and} \qquad \lim_{\delta\downarrow 0} |I \cap [\,-\delta, 0\,]|/\delta = 1.$$

Thus we can treat the right and the left neighbourhood of 0 separately. We can now apply lemma 16 to the function $\lambda(\beta)$ instead of $\varphi(x)$, and to the sets $I_+ \cup \{0\}$ and $I_- \cup \{0\}$ instead of X. Observe from the definition (14) that Λ is closed. Let us show that the condition (41) is fulfilled as well. Assume that $\beta_1, \beta_2 \in I$, $|\beta_1| < |\beta_2|$. Then $(\beta_1, \lambda(\beta_1))$, $(\beta_2, \lambda(\beta_2))$ and $(\beta_1, \lambda(\beta_2))$ belong to the domain of G. Write

$$\lambda(\beta_1) - \lambda(\beta_2) = G(\beta_1, \lambda(\beta_1)) - G(\beta_1, \lambda(\beta_2)) + G(\beta_1, \lambda(\beta_2)) - G(\beta_2, \lambda(\beta_2))$$

and use lemma 13 to obtain

$$\lambda(\beta_1) - \lambda(\beta_2) = \frac{G(\beta_1, \lambda(\beta_2)) - G(\beta_2, \lambda(\beta_2))}{1 + \langle g(\beta_1, \lambda(\beta_1)),\ g(\beta_1, \lambda(\beta_2))\rangle}.$$

Deduce from (34) that

$$0 < \langle g(\beta_1, \lambda(\beta_1)),\ g(\beta_1, \lambda(\beta_2))\rangle = \mathrm{O}(|\beta_2|^2) \qquad \text{as } |\beta_1| \leqslant |\beta_2| \to 0$$

and note that (38) can be rewritten as

$$|\partial_{\beta^2} G(\beta, \lambda) - G_2(\lambda)| \leqslant |G_2(0)|/2.$$

One readily concludes that there exist constants $0 < A \leqslant B$ and a bound $\beta_\star > 0$ such that

$$A|\beta_1^2 - \beta_2^2| \leqslant |\lambda(\beta_1) - \lambda(\beta_2)| \leqslant B|\beta_1^2 - \beta_2^2| \qquad \text{for all } \beta_1, \beta_2 \in I \cap [-\beta_\star, \beta_\star].$$

Lemma 17. 0 is a point of density for the set I.

Now we can approach the problem of the asymptotic series. Consider first the following situation. Let $\{H_k\}_{k=0}^\infty$ be a sequence of complex meromorphic functions such that 0 is a regular point for all of them and, moreover, $H_0(0) = 0$, $H_0'(0) \neq 0$. Then

$$\Phi(x, y) := \sum_{k=0}^{\infty} x^k H_k(y) \in \mathbb{C}[[x, y]]$$

is well defined as a formal power series in x and y. Denote by $\varphi^f(x) \in \mathbb{C}[[x]]$ the solution to the problem

$$\varphi^f(0) = 0 \qquad \text{and} \qquad \Phi(x, \varphi^f(x)) = 0$$

which exists and is unique in the class of formal power series. Set

$$\mathcal{R}_\Phi := \mathbb{C} \setminus \bigcup_{k=0}^{\infty} \{\text{the poles of the function } H_k\}$$

and let $R(y)$ be the radius of convergence of the series $\Phi(x, y)$ in the variable x, with $y \in \mathcal{R}_\Phi$ being fixed.

Lemma 18. Let φ be a complex function defined on $X \subset \mathbb{C}$ and assume that:

(i) $0 \in X$ is an accumulation point of X,
(ii) $\forall x \in X$, $|x| < R(\varphi(x))$ (and so the value $\Phi(x, \varphi(x))$ is well defined),
(iii) φ solves the problem

$$\varphi(0) = 0 \qquad \text{and} \qquad \Phi(x, \varphi(x)) = 0 \quad \forall x \in X$$

(iv) there exists $\mu > 0$ such that

$$\Phi_N(x, \varphi(x)) = \mathrm{O}(|x|^{\mu(N+1)}) \qquad \forall N \in \mathbb{Z}_+$$

$$\Phi_N(x, y) := \sum_{k=0}^{N} x^k H_k(y).$$

Then $\varphi^f(x)$ is an asymptotic series for $\varphi(x)$.

Proof. Denote by φ_M^f the truncation of order M of φ^f (thus φ_M^f is a polynomial of degree at most M and $\varphi^f(x) - \varphi_M^f(x) \in x^{M+1}\,\mathbb{C}[[x]]$). We have to show that

$$\varphi(x) - \varphi_M^f(x) = \mathrm{O}(|x|^{M+1}) \qquad \forall\, M \in \mathbb{Z}_+.$$

Denote by $\varphi^{(N)}(x)$ the unique solution to the problem

$$\varphi^{(N)}(0) = 0 \qquad \text{and} \qquad \Phi_N(x, \varphi^{(N)}(x)) = 0$$

in the class of germs of holomorphic functions at $x = 0$. Clearly,

$$\varphi_M^f(x) = \varphi_M^{(N)}(x) \qquad \text{if } \; N \geqslant M.$$

Note that the requirement (4), with $N = 0$, means that $H_0(\varphi(x)) = \mathrm{O}(|x|^\mu)$. Since $H_0'(0) \neq 0$ we find that $\lim_{x\to 0} \varphi(x) = 0$. Obviously, it also holds that $\lim_{x\to 0} \varphi^{(N)}(x) = 0$. Observe that $\partial_y \Phi_N(0,0) = H_0'(0) \neq 0$. Consequently, for any $n \in \mathbb{Z}_+$, there exist positive constants c_N, δ_N such that

$$|\Phi_N(x, \varphi(x)) - \Phi_N(x, \varphi^{(N)}(x))| \geqslant c_N\, |\varphi(x) - \varphi^{(N)}(x)| \qquad \forall\, x \in X \quad |x| \leqslant \delta_N.$$

Fix $M \in \mathbb{Z}_+$ and choose $N \in \mathbb{Z}_+$ such that $N \geqslant M$ and $\mu(N+1) \geqslant M+1$. Write

$$\varphi(x) - \varphi_M^f(x) = \varphi(x) - \varphi^{(N)}(x) + \varphi^{(N)}(x) - \varphi_M^{(N)}(x) = \varphi(x) - \varphi^{(N)}(x) + \mathrm{O}(|x|^{M+1}).$$

On the other hand

$$\begin{aligned} c_N\, |\varphi(x) - \varphi^{(N)}(x)| &\leqslant |\Phi_N(x, \varphi(x)) - \Phi_N(x, \varphi^{(N)}(x))| = |\Phi_N(x, \varphi(x))| \\ &= \mathrm{O}(|x|^{\mu(N+1)}). \end{aligned}$$

We conclude that $\varphi(x) - \varphi_M^f(x) = \mathrm{O}(|x|^{M+1})$, as required. □

Lemma 17 is directly applicable to the function $\Phi(\beta, \lambda) := G(\beta, \lambda) - \lambda$ and to our solution $\lambda(\beta)$.

Lemma 19. The formal power series $\sum_{M=0}^{\infty} \xi_M\, \beta^M$, with ξ_M given in (10) and (13), is an asymptotic series for the function $\lambda(\beta)$ defined on I.

In summary let us state that lemmas 14, 17 and 19 jointly verify the existence and the properties of the function $\lambda(\beta)$ and thus the proof of proposition 1 has been completed.

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